Assignment-3

$f_c = 80 Hz$

$f_s = 200 Hz$

$N = 2$

Normalized cut off angular frequency, $\omega_c = \frac{f_c}{f_s} 2\pi$

$= \frac{80}{200} \times 2\pi$

$= 0.8\pi$ radians/s

So, $T = 1$

The prewarped s-domain cut off frequency,

$\Omega_c = \frac{2}{T} \tan\left(\frac{\omega_c}{2}\right)$

$= 2 \tan(0.4\pi)$

$= 2 \times 3.0777$

$= 6.1554$ radians/s

We know,

$$S_k = \Omega_c \exp\left[j\pi\left(\frac{1}{2} + \frac{2k-1}{2N}\right)\right], \quad k = 1, 2, \ldots, 2N$$

So, $s_1 = 6.1554 \exp\left[j\pi\left(\frac{1}{2} + \frac{1}{4}\right)\right]$

$= 6.1554 \exp(j0.75\pi)$

$= 6.1554 \left[\cos(0.75\pi) + j\sin(0.75\pi)\right]$

$= -4.3525 + j4.3525$

$$S_2 = 6.1554 \exp\left[j\pi\left(\frac{1}{2}+\frac{3}{4}\right)\right]$$

$$= 6.1554 \exp(j1.25\pi)$$

$$= -4.3525 - j4.3525$$

$$S_3 = 6.1554 \exp\left[j\pi\left(\frac{1}{2}+\frac{5}{4}\right)\right]$$

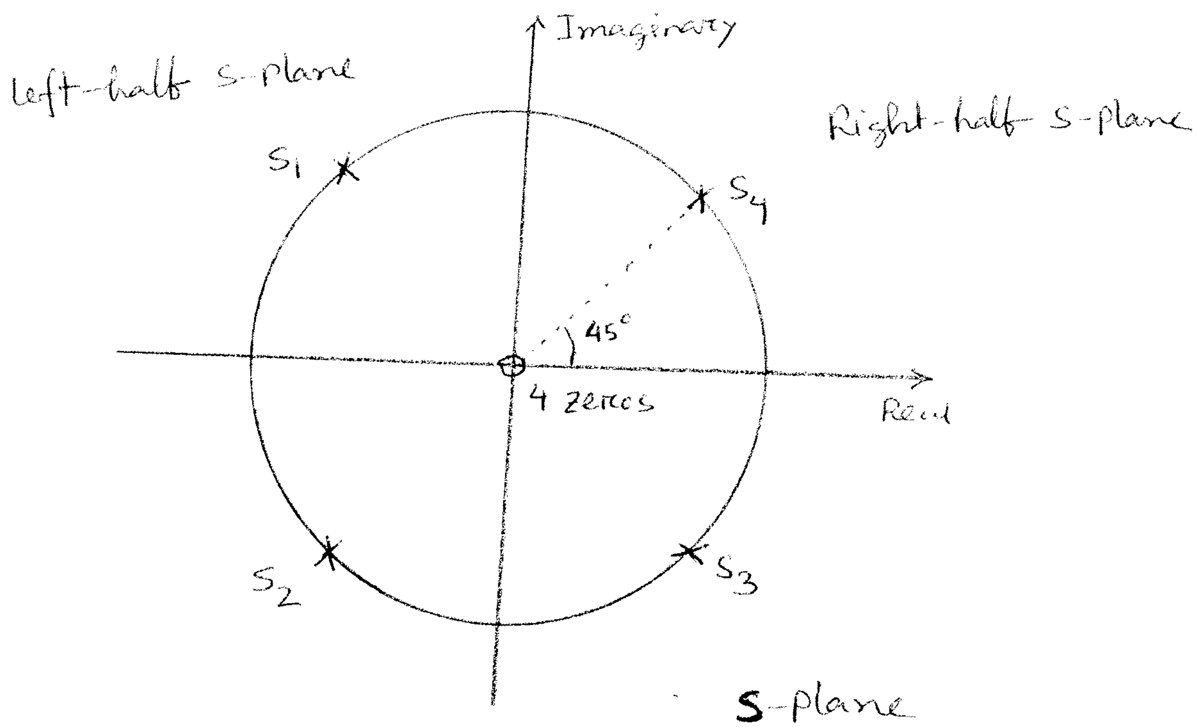
$$= 6.1554 \exp(j1.75\pi)$$

$$= 4.3525 - j4.3525$$

$$S_4 = 6.1554 \exp(j2.25\pi)$$

$$= 4.3525 + j4.3525$$

The pole-zero diagram will be,



Only s_1 and s_2 poles are on the ~~left hand side~~ left-half of the s plane. So, the transfer function of the filter is,

$$H_a(s) = \frac{G}{(s+4.3525-j4.3525)(s+4.3525+j4.3525)}$$

where G is the gain factor.

$$H_a(s) = \frac{G}{s^2+8.705s+37.8885}$$

Normalizing for unity gain at DC $(s=0)$, the gain factor,

$$G = \frac{1}{H_a(s)|_{s=0}} = \frac{1}{\frac{1}{0+0+37.8885}}$$

$$= 37.8885$$

So, the normalized filter in Laplace domain,

$$H_a(s) = \frac{37.8885}{s^2+8.705s+37.8885}$$

The bilinear transformation

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$= 2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

Applying bilinear transformation,

$$H(z) = \frac{37.8885}{4\frac{(1-z^{-1})^2}{(1+z^{-1})^2} + 8.705 \times 2\frac{(1-z^{-1})}{(1+z^{-1})} + 37.8885}$$

$$= \frac{37.8885(1+z^{-1})^2}{4(1-z^{-1})^2 + 17.41(1-z^{-1})(1+z^{-1}) + 37.8885(1+z^{-1})^2}$$

$$= \frac{37.8885(1+2z^{-1}+z^{-2})}{4(1-2z^{-1}+z^{-2}) + 17.41(1-z^{-2}) + 37.8885(1+2z^{-1}+z^{-2})}$$

$$= \frac{37.8885(1+2z^{-1}+z^{-2})}{4-8z^{-1}+4z^{-2}+17.41-17.41z^{-2}+37.8885+75.777z^{-1}+37.8885z^{-2}}$$

$$= \frac{37.8885(1+2z^{-1}+z^{-2})}{59.2985+67.777z^{-1}+24.4785z^{-2}}$$

$$= \frac{0.6389(1+2z^{-1}+z^{-2})}{1+1.1430z^{-1}+0.4128z^{-2}}$$

For unity gain at DC ($z=1$);

$$H(z)\Big|_{z=1} = \frac{0.6389(1+2+1)}{1+1.1430+0.4128}$$

$$= \frac{2.5556}{2.5558}$$

$$\approx 1$$

So, the filter $H(z)$ is already normalized for unity gain at dc.

The Filter coefficients are,

$$[a] = [1 \quad 1.1430 \quad 0.4128]$$

$$[b] = [0.6389 \quad 1.2778 \quad 0.6389]$$

The pole-zero plot

$$H(z) = \frac{0.6389(1+z^{-1})^2}{1 + 1.1430z^{-1} + 0.4128z^{-2}}$$

The zeros are, $0.6389(1+z^{-1})^2 = 0$

$$\Rightarrow (z+1)^2 = 0$$

$$z_1 = -1, \quad z_2 = -1$$

~~The poles are,~~

$$\cancel{p_1 = -1.1430 + \sqrt{(1.1430)^2 - 4}}$$

For the poles,

$$1 + 1.1430z^{-1} + 0.4128z^{-2} = 0$$

$$\Rightarrow z^2 + 1.1430z + 0.4128 = 0$$

$$p_1 = \frac{-1.1430 + \sqrt{(1.1430)^2 - 4 \times 1 \times 0.4128}}{2 \times 1}$$

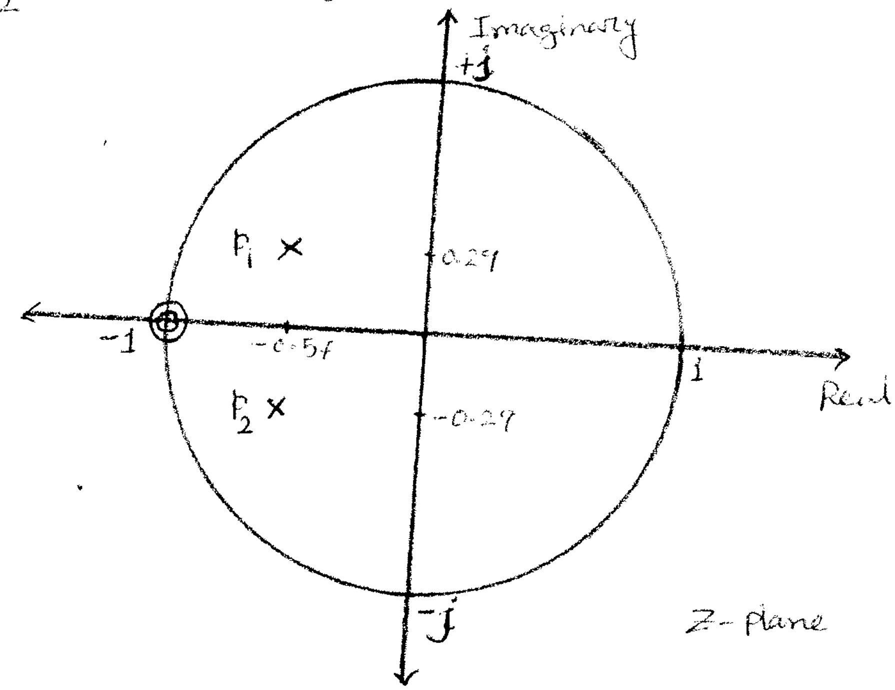
$$= \frac{-1.1430 + \sqrt{-0.3448}}{2}$$

$$= \frac{-1.1430 + 0.5872j}{2}$$

$$= -0.5715 + j0.2936$$

Similarly,

$p_2 = -0.5715 - j0.2936$



pole-zero plot

MATLAB

Butter-worth filter coefficients

$[b] = [0.6389 \quad 1.2779 \quad 0.6389];$

$[a] = [1 \quad 1.1430 \quad 0.4128];$

They are the same as calculated.

The frequency response of the filter is attached.

Fig: Frequency response of the filter